

Space curves: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$

def: lim of space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
 $t \rightarrow t_0$ is continuous if they all exist

helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \rightarrow$ circle in xy plane

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \rangle$$

$$\vec{r}(t) = \langle 14 + \sin(20t + \cos t), 4 + \sin(20t + \sin t), (\cos 2t) \rangle$$

ex: $\lim_{t \rightarrow \infty} \langle \frac{1+t^2}{1-t}, \arctan(t), \frac{1-e^{-2t}}{t} \rangle$

Sol: $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{1+t^2}{1-t} = \lim_{t \rightarrow \infty} \frac{\frac{1}{t^2} + 1}{\frac{1}{t} - 1} = \frac{0+1}{0-1} = -1$

Continuity:
a space curve
 $\vec{r}(t)$ is cont.
at $t=a$ when
 $\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \arctan(t) = \frac{\pi}{2}$$

$$\lim_{t \rightarrow \infty} z(t) = \lim_{t \rightarrow \infty} \frac{1-e^{-2t}}{t} = \lim_{t \rightarrow \infty} \frac{2e^{-2t}}{1} = \lim_{t \rightarrow \infty} 2e^{-2t} = 0$$

resultant vector is $\langle -1, \frac{\pi}{2}, 0 \rangle$

where's
ex: $\vec{r}(t) = \langle \frac{1+t^2}{1-t}, \arctan(t), \frac{1-e^{-2t}}{t} \rangle$ continuous?

$\vec{r}(t)$ is continuous at a
iff each component is

domains: $x(t): t \neq \pm 1 \Rightarrow (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$, $y(t): (-\infty, \infty)$

derivative
of space curve $\vec{r}(t)$ $\vec{r}(t)$ is cts on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

at $t=a$ is $\vec{r}'(a) = \lim_{h \rightarrow 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$ ← extremely important

ex: $\vec{r}'(t)$ for $\vec{r}(t) = \langle t, t^2, \sqrt{t} \rangle$

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \langle t+h, (t+h)^2, \sqrt{t+h} - \sqrt{t} \rangle$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \langle h, 2t+h, \sqrt{t+h} - \sqrt{t} \rangle = \lim_{h \rightarrow 0} \langle 1, 2t+h, \frac{\sqrt{t+h} - \sqrt{t}}{h} \rangle$$

$$= \lim_{h \rightarrow 0} \langle 1, \lim_{h \rightarrow 0} (2t+h), \lim_{h \rightarrow 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \rangle = \langle 1, 2t, \frac{1}{2\sqrt{t}} \rangle$$

$$= \lim_{h \rightarrow 0} \frac{(t+h) - t}{h(\sqrt{t+h} + \sqrt{t})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{2\sqrt{t}} \quad \text{what really happened?}$$

$$\lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \left\langle \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}, \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h} \right\rangle = \langle x'(t), y'(t) \rangle$$

derivatives are \odot
computed componentwise!

addresses instantaneous velocity in each direction

Properties: $\vec{r}(t)$ and $\vec{s}(t)$ are space curves, $c(t)$ is a scalar fn, derivatives exist

$$\textcircled{1} \frac{d}{dt} [\vec{r}(t) + \vec{s}(t)] = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}$$

$$\textcircled{2} \frac{d}{dt} [c(t) \vec{r}(t)] = c'(t) \vec{r}(t) + c(t) \vec{r}'(t)$$

$$\textcircled{3} \frac{d}{dt} [\vec{r}'(t) \cdot \vec{s}'(t)] = \frac{d}{dt} [\langle x'(t), y'(t) \rangle \cdot \langle a'(t), b'(t) \rangle] = \frac{d}{dt} [x'(t)a'(t) + y'(t)b'(t)]$$

$$= (x''(t)a'(t) + y''(t)b'(t)) + (x'(t)a''(t) + y'(t)b''(t)) = \langle x'', y'' \rangle \cdot \langle a', b' \rangle + \langle x', y' \rangle \cdot \langle a'', b'' \rangle$$

$$\Rightarrow \vec{r}''(t) \cdot \vec{s}'(t) + \vec{r}'(t) \cdot \vec{s}''(t)$$

$$\textcircled{4} \frac{d}{dt} [\vec{r}(t) \times \vec{s}(t)] = \vec{r}'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}'(t)$$

do not switch around

$$\textcircled{5} \frac{d}{dt} [\vec{r}(c(t))] = \vec{r}'(c(t)) \cdot c'(t) \leftarrow \text{chain rule}$$

exercise: verify \uparrow for \mathbb{R}^3 curves

~~unit tangent vector~~ $\vec{r}'(t)$ = tangent vector to $\vec{r}(t)$ at time t

unit tangent vector is $\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ $\vec{r}'(t) \neq 0$ speed of $\vec{r}(t)$ is $|\vec{r}'(t)|$

exercise: Prove that if $\vec{r}(t)$ has constant speed, then $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal

integrals of space curves:

$$\int_a^b \vec{r}(t) dt = \lim_{n \rightarrow \infty} \sum_{k=1}^n \vec{r}(t_k^*) \Delta t_k$$

$$\text{let } \int_a^b \vec{r}(t) dt = \left\langle \int_a^b x(t) dt, \int_a^b y(t) dt, \int_a^b z(t) dt \right\rangle \text{ for } \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

are lengths!
Pick lots of points and
add lengths

interpretation: $\int_a^b \vec{r}(t) dt$ represents displacement

Approximate w/ straight lines

These approximations limit to tangent line $\textcircled{2}$ Sum of lines' lengths is known
are length $L(a,b)$ of $\vec{r}(t) = \int_a^b |\vec{r}'(t)| dt$